On the Minimal Model of Anyons

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Abstract

We present new geometric formulations for the fractional spin particle models on the minimal phase spaces. New consistent couplings of the anyon to background fields are constructed. The relationship between our approach and previously developed anyon models is discussed.

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1 Introduction

Particles with fractional statistics, so called anyons [1, 2], living in $(1+2)$–dimensional space–time have attracted a considerable interest in recent years. This interest basically ensues from the important role which the anyons were established to play in some planar physics phenomena, like the fractional quantum Hall effect [3], high–T_c superconductivity [2] as well as in some $d = 3$ quantum field models [4-6].

The most popular approach to the description of anyons is to employ the Chern–Simons gauge field to generate statistics of the particles [6-9]. Up to now, however, it is not quite clear whether the fractional statistics states may emerge in this approach as the result of interaction with the gauge field only or this is an inherent quality of the particles themselves. Even in the latter case, it would not be possible to eliminate the Chern–Simons field without violation of locality [6].

Another traditional method to describe anyons, which we follow in the present paper, is the group–theoretical approach [10-20]. This implies to derive the quantum theory by quantizing a classical mechanics model based on some Poincaré–invariant Lagrangian. Within this approach the possibility for the particle spin to be fractional is made evident from the fact that the universal covering map for the $(1+2)$–dimensional Poincaré group is infinite-sheeted.

In constructing the mechanics models of spinning particles, the phase space is usually extended by some internal variables destined to realize the spin part of the Lorentz generators (see, e.g., [21] and references therein). At that an appropriate set of constraints should be imposed on the phase space variables in order to obtain the irreducible dynamics of spinning particle with correct number of degrees of freedom. In particular, the particle mass and spin are to be identically conserved on the constrained surface. Of especial interest are minimal models defined to contain a minimal number of constraints providing the identical conservation of the Casimir functions associated with the phase space Poincaré generators. Such models are, in a sense, universal since any correct spinning particle model should turn into a minimal one upon restricting the dynamics to a surface of constraints. Another important peculiarity of the models consists in strong restrictions the requirement of minimality imposes on the topological structure of the phase space. In particular, its dimension proves to be uniquely determined from a simple counting of degrees of freedom.

For $d = 3$ any minimal model is seen to be equivalent to a dynamical system with two first-class constraints corresponding to the Casimir operators of the Poincaré group $ISO(1, 2)$. Since for $d = 3$ the spinning particle possesses the same number of degrees of freedom as a spinless one, the internal (spin) sector of the phase space has to be two-dimensional. A useful classification of $d = 3$ minimal models has been recently given [19]. It is based on the observation that the phase space functions $M^a M_a$, M_a being the spin part of the Lorentz generators, commute with the Poincaré generators, with respect to the Poisson bracket, hence

$$M^a M_a = C = const \tag{1.1}$$

is a constant parameter of the model. If M_a are realized in the form $M_a = M_a(z_i)$, where z_i, $i = 1, 2$, are internal phase space variables, then Eq. (1.1) implies that the topology of the spin space is determined by the value of C. For $C < 0$ we have a two-sheeted hyperboloid, for $C > 0$ a one-sheeted hyperboloid which turns into a cone for $C = 0$. It has been shown [19] how to realize the minimal model for each value of C by describing the spin degrees of freedom in terms of the constrained variables M_a required to form the Lorentz algebra with respect to the Poisson bracket.

In the present paper we show that all minimal models uniquely originate from natural physical and geometrical principles. The starting points of our approach are as follows. First, the spin space is the cotangent bundle $T^*(S^1)$ of a one-sphere for $C \geq 0$ and coincides with Lobachevsky space $\mathcal{L}$ for $C < 0$. Second, the action functional should be specified in terms of geometric invariants related to the phase space $T^*(\mathbf{R}^{1,2} \times S^1)$ for $C \geq 0$ and $T^*(\mathbf{R}^{1,2}) \times \mathcal{L}$ for $C < 0$. Our approach allows to obtain new insights into the structure of the minimal models. In particular, we construct new consistent couplings of the anyon to background field. We also demonstrate canonical equivalence of some models previously considered to describe different quantum dynamics.

There exist several general approaches to the quantization of particles with fractional spin. In the framework of the group–theoretical approach [10,17-19] one should use the infinite-dimensional unitary representations of the universal covering group $\overline{SL(2, \mathbf{R})}$. Such representations arise quite naturally in our consideration, if to realize the Hilbert space of physical states as an appropriate function space on the classical phase manifold. We demonstrate this statement in detail for the case of $C \geq 0$ in subsec. 3.2.

The paper is organized as follows. Section 2 provides a brief description of $\mathcal{L}$ and S^1 as homogeneous spaces of $ISO(1, 2)$ in the form most appropriate for our construction. The consideration follows the general lines used in Ref. [22] where a similar description has been given for the action of $ISO(1, 3)$ on sphere S^2.

In section 3 we give the Hamilton and Lagrange formulations for the minimal anyon model with the configuration manifold $\mathcal{M}^4 = \mathbf{R}^{1.2} \times S^1$. The minimal model with the phase space $T^*(\mathbf{R}^{1,2}) \times \mathcal{L}$ is described in section 4. For the latter model we show that in the special case, when the particle momentum is parallel to the spin vector, the two independent first-class constraints of the theory split into three ones including second-class constraints.

In section 5 we discuss the problem of interaction with external fields in the framework of the minimal anyon models. We begin with establishing the equivalence between our model in the special case described and the well known anyon models [11-13,19] based on the use of the Dirac monopole symplectic two-form. Thus, along with the latter class of particle models [11, 12], the minimal model in the special case admits the consistent coupling to an arbitrary week external fields. In general position, however, the momentum and the spin vector are not parallel to each other and the minimal model contains essentially two first-class constraints. Of course, such a structure of constraints is not compatible with arbitrary background. Nev-

ertheless, the model is shown to admit the interaction with an electromagnetic field subject to the free Maxwell equations with Chern–Simons term as well as with a constant curvature gravitational background. The restrictions on the fields appear from the compatibility conditions for the anyon model gauge symmetries. The similar phenomenon is known for some superparticle models [28].

2 The Poincaré group action on S^1 and $\mathcal{L}$

In this section we describe the action of $ISO(1,2)$ on S^1 and $\mathcal{L}$ in the form adapted for further consideration. Let us begin with recalling the realizations of these manifolds as homogeneous spaces of $SO^\uparrow(1,2)$. It is useful to consider S^1 and $\mathcal{L}$ as submanifolds of the Riemann sphere $\mathbf{CP}^1 = \mathbf{C} \bigcup \{\infty\}$. We identify the Lobachevsky plane with the upper half-plane

$$\mathcal{L}^{(+)} = \{z \in \mathbf{C}, \quad \mathrm{Im}\, z > 0\} \tag{2.1}$$

or the down half-plane

$$\mathcal{L}^{(-)} = \{z \in \mathbf{C}, \quad \mathrm{Im}\, z < 0\} \tag{2.2}$$

of $\mathbf{C}$, and realize S^1 as $\{z \in \mathbf{R} \bigcup \{\infty\}\}$, thus having $\mathbf{CP}^1 = \mathcal{L}^{(+)} \bigcup S^1 \bigcup \mathcal{L}^{(-)}$. We use both the angle (φ) and stereographic ($z = \mathbf{ctg}\, \frac{\varphi}{2}$) parametrizations of S^1, $\varphi \in [\,0\,,2\pi\,]$.

The Lorentz group $SO^\uparrow(1,2) \cong SL(2,\mathbf{R})/Z_2$ acts on $\mathbf{CP}^1$ by fractional linear transformations

$$N: \quad z \to z' = \frac{az - b}{d - cz} \quad, \tag{2.3}$$

$$N \equiv \left\| N_\alpha{}^\beta \right\| = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{R})$$

With respect to this action $\mathbf{CP}^1$ consists of three orbits: $\mathcal{L}^{(+)}$, S^1 and $\mathcal{L}^{(-)}$.

One can bring Eq. (2.3) to a manifestly covariant form introducing (by analogy with the four–dimensional case [22]) two-component objects $z^\alpha \equiv (1, z)$ and $z_\alpha \equiv \epsilon_{\alpha\beta} z^\beta = (-z, 1)$ transforming under (2.3) by the law

$$N: \quad z^\alpha \to z'^\alpha = \left(\frac{\partial z'}{\partial z}\right)^{1/2} z^\beta N^{-1}{}_\beta{}^\alpha \qquad \bar{z}^\alpha \to \bar{z}'^\alpha = \left(\frac{\partial \bar{z}'}{\partial \bar{z}}\right)^{1/2} \bar{z}^\beta N^{-1}{}_\beta{}^\alpha, \tag{2.4}$$

or in the infinitesimal form

$$\delta z = \omega_{\alpha\beta} z^\alpha z^\beta \qquad \delta\bar{z} = \omega_{\alpha\beta} \bar{z}^\alpha \bar{z}^\beta, \tag{2.5}$$

where $\omega_{\alpha\beta}$ are the parameters of Lorentz transformations.

The above relations imply that the Lorentz generators of scalar fields look like

$$M_a = -i\xi_a \partial_z - i\bar{\xi}_a \partial_{\bar{z}} \tag{2.6}$$

on $\mathcal{L}$ and

$$M_a = -i\zeta_a \partial_z = -in_a \partial_\varphi \tag{2.7}$$

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and on S^1 (z being real on S^1). Here[2]

$$\xi_a \equiv -\frac{1}{2}(\sigma_a)_{\alpha\beta} z^\alpha z^\beta = -\frac{1}{2}(1+z^2, 1-z^2, 2z)$$

$$\zeta_a \equiv -\frac{1}{2}(\sigma_a)_{\alpha\beta} z^\alpha \bar{z}^\beta = -\frac{1}{2}(1+z\bar{z}, 1-z\bar{z}, z+\bar{z}) \tag{2.8}$$

$$n_a \equiv (1, -\cos\varphi, \sin\varphi) \ ,$$

where ∂_z, $\partial_{\bar{z}}$ and ∂_φ denote the partial derivatives with respect to $z, \bar{z}$ and φ. The three-dimensional Dirac matrices σ_a are chosen to be real and symmetric

$$(\sigma_0)_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (\sigma_1)_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (\sigma_2)_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.9}$$

$$(\sigma_a)_{\alpha\beta}(\sigma^a)_{\gamma\delta} = -(\epsilon_{\alpha\gamma}\epsilon_{\beta\delta} + \epsilon_{\alpha\delta}\epsilon_{\beta\gamma}) \ .$$

The only Lorentz-invariant objects, constructed in terms of z^α and $\bar{z}^\alpha$, are the Kähler metric on $\mathcal{L}$ and the associated two-form

$$\mathrm{d}s^2 = 4\frac{\mathrm{d}z\mathrm{d}\bar{z}}{\chi^2} \qquad \boldsymbol{\Sigma} = -2i\frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{\chi^2} \ , \tag{2.10}$$

where $\chi \equiv \epsilon_{\alpha\beta} z^\alpha \bar{z}^\beta = z - \bar{z}$. There are no internal Lorentz-invariant structures on S^1. Both $\mathcal{L}$ and S^1 admit external invariants. In particular, let p^a be a time-like vector. Then the combination

$$\mathrm{d}\sigma \equiv \left| \frac{\mathrm{d}z}{(p,\zeta)} \right|$$

remains unchanged under the Lorentz transformations. For $z = \bar{z}$ we also have

$$\mathrm{d}\sigma = \frac{\mathrm{d}\varphi}{(p,n)} \tag{2.11}$$

and $\mathrm{d}\sigma$ can be treated as a Lorentz-invariant extension of arc length (in a rest reference system where $p^a \sim (1,0,0)$ we have $\mathrm{d}\sigma \sim \mathrm{d}\varphi$). As we shall show, this invariant appears in the mechanics Lagrangian on $\mathbf{R}^{1,2} \times S^1$ along with the Minkowski interval $(-dx^a dx_a)^{1/2}$. Associated with p^a are a number of invariants on $\mathcal{L}$, for instance $(p,\zeta)/\chi$ and $\mathrm{d}z/(p,\xi)$.

The action of $SO^\uparrow(1,2)$ on $\mathbf{CP}^1$ can be extended to that of the Poincaré group by defining the translations to act as the identity map of $\mathbf{CP}^1$. As for discrete Lorentz transformations, the mappings on $\mathbf{CP}^1$

$$z \rightarrow -\bar{z} \qquad \left(z \rightarrow -1/\bar{z}\right) \tag{2.12a}$$

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[2]We use Latin letters to denote vector indices and Greek letters for spinor ones; the space–time metric signature is chosen to be $(-,+,+)$; the spinor indices are raised and lowered with the use of the spinor metric $\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha} = -\epsilon_{\alpha\beta}$ ($\alpha, \beta = 0, 1$), $\epsilon^{01} = 1$ by the rule $\psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta, \psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta$.

can be identified with the space and time inversions of $\mathbf{R}^{1,2}$

$$(x^0, x^1, x^2) \to (x^0, x^1, -x^2) \quad (x^0, x^1, x^2) \to (-x^0, x^1, x^2) \tag{2.12b}$$

respectively. Under the discrete Poincaré transformations, S^1 transforms into itself, while $\mathcal{L}^{(+)}$ turns into $\mathcal{L}^{(-)}$ and vice versa. Hence $\mathbf{CP}^1$ consists of two orbits of the Poincaré group, that is, S^1 and $\mathcal{L} = \mathcal{L}^{(+)} \bigcup \mathcal{L}^{(-)}$.

3 Anyon model on $\mathcal{M}^4$

3.1 Classical dynamics

In accordance with the analysis of Sec. 1, the phase space of a minimal anyon model should be an eight-dimensional transformation space of the Poincaré group. Let us suppose also that the phase space can be realized as the cotangent bundle of some manifold $\mathcal{M}^4$. Then $\mathcal{M}^4$ proves to have the unique form $\mathbf{R}^{1,2} \times S^1$. In this section we construct the particle dynamics on $\mathcal{M}^4$.

Let $A(x^a, p_a, \varphi, p_\varphi)$ and $B(x^a, p_a, \varphi, p_\varphi)$ be scalar functions on $T^*(\mathcal{M}^4)$. In terms of the Poisson bracket

$$\{A, B\} = \frac{\partial A}{\partial x^a}\frac{\partial B}{\partial p_a} + \frac{\partial A}{\partial \varphi}\frac{\partial B}{\partial p_\varphi} - (A \leftrightarrow B). \tag{3.1}$$

an infinitesimal Poincaré transformation reads

$$\delta A = \{A, -f^a P_a + \omega^a J_a\},$$

where f^a and ω^a are the parameters of translations and Lorentz rotations respectively. The Hamiltonian generators are given by

$$P_a = p_a \qquad J_a = -\epsilon_{abc}x^b p^c + M_a \qquad M_a = n_a p_\varphi, \tag{3.2}$$

$\epsilon_{012} = 1$, n_a is defined in (2.8). The particle dynamics is governed by the first–class constraints

$$P^2 + m^2 = p^2 + m^2 \approx 0, \tag{3.3}$$

$$W = (P, M) = (p, n)p_\varphi \approx ms, \tag{3.4}$$

expressing the identical conservation of the mass m and spin s of anyon. For the sake of reparametrization invariance, the Hamiltonian should be a linear combination of constraints, hence the action looks like

$$S = \int d\tau \{p_a \dot{x}^a + p_\varphi \dot{\varphi} - \frac{e(\tau)}{2}(p^2 + m^2) - \lambda(\tau)((p, n)p_\varphi - ms)\}, \tag{3.5}$$

where $e(\tau)$ and $\lambda(\tau)$ are Lagrange multipliers, the dots denote derivatives with respect to evolution parameter τ. Since the constraint (3.4) is linear in the angle momentum p_φ, it can be readily solved thus resulting in the action functional

$$S = \int d\tau \{ p_a \dot{x}^a + \frac{ms}{(p,n)} \dot{\varphi} - \frac{e(\tau)}{2}(p^2 + m^2) \} \,. \tag{3.6}$$

The action contains all independent worldline Poincaré invariants on $\mathcal{M}^4$. However our manifold also admits the one-form

$$\boldsymbol{\Omega} = \frac{(p, \partial_\varphi n)}{(p, n)} d\varphi \,, \tag{3.7}$$

that changes under the Poincaré transformations only by exact contributions, $\delta\boldsymbol{\Omega} = d(\partial_\varphi n_a)\omega^a$. This implies that the one-form is allowable in the action functional

$$S = \int d\tau \{ p_a \dot{x}^a + \frac{ms}{(p,n)} \dot{\varphi} + \varrho \frac{(p, \partial_\varphi n)}{(p, n)} \dot{\varphi} - \frac{e(\tau)}{2}(p^2 + m^2) \} \,, \tag{3.8}$$

where ϱ is a real parameter. This functional changes by boundary terms under the Lorentz transformations in contrast to S (3.6), the latter being a genuine invariant. As a consequence, the Noether currents J_a (3.2) take the modified form

$$\tilde{J}_a = -\epsilon_{abc}x^b p^c + \tilde{M}_a \,, \quad \tilde{M}_a = n_a p_\varphi - \varrho \partial_\varphi n_a \,, \tag{3.9}$$

and the constraint (3.4) is modified by the rule

$$P^a \tilde{M}_a = (p,n)p_\varphi - \varrho(p, \partial_\varphi n) \approx ms \,. \tag{3.10}$$

It is essential that the constraint (3.10) expresses the same physical content as (3.4) has done before, i.e. the strong conservation law of the anyon spin s. This fact is not surprising in so far as one can show that the models (3.6) and (3.8) are related to each other by some canonical transformation with the generating function

$$\mathcal{F}(x^a, \varphi, \tilde{p}_a, \tilde{p}_\varphi) = \varrho \int_{\varphi_0}^{\varphi} \frac{(\tilde{p}, \partial_\varphi n)}{(\tilde{p}, n)} d\varphi + \varphi \tilde{p}_\varphi + x^a \tilde{p}_a \,, \tag{3.11}$$

where the variables without tilde are related to the model (3.6), whereas those with tilde – to (3.8), and φ_0 is an arbitrary constant. The interpretation of the parameter ϱ is also obvious. From (3.9) it follows that the value of ϱ fixes the value of squared spin $\tilde{M}_a \tilde{M}^a = \varrho^2$ (thus the spinning momentum vector is not time-like).

Let us turn to the Lagrangian formulation of the model. The momenta and the Lagrange multiplier can be eliminated from (3.8) by making use of the mass shell constraint and the equations of motion. As a result, one obtains the following Lagrangian

$$L = -m\sqrt{-\dot{x}^2 \left(1 - \frac{2s}{m}\frac{\dot{\varphi}}{(\dot{x}, n)} - \frac{\varrho^2}{m^2}\frac{\dot{\varphi}^2}{(\dot{x}, n)^2} \right)} + \varrho\frac{(\dot{x}, \partial_\varphi n)}{(\dot{x}, n)}\dot{\varphi} \,. \tag{3.12}$$

In the limit $s \to 0$, $\varrho \to 0$ L reduces to the Lagrangian of a spinless particle.

The geometric anyon model on $\mathcal{M}^4$ with $\varrho \neq 0$ can be regarded as a reduction to $1 + 2$ dimensions of the $(1 + 3)$-dimensional spinning particle model suggested in Ref. [24]. The latter model contains two parameters analogous to ϱ and s and possesses an interesting property. For special relation between ϱ and s the structure of constraints is radically altered: a first class constraint splits into pair of second-class constraints (what may be relevant in the framework the problem of coupling to external fields). In three dimensions, however, $\mathcal{M}^4$ does not admit similar phenomenon. Moreover, for all values of ϱ the corresponding models are canonically equivalent.

Let us discuss the dynamics in the model. The equations of motion in Minkowski space can be written in the form

$$\dot{p}_a = 0 \qquad \dot{x}^a = e p^a + m s \frac{n^a}{(p, n)^2} \dot{\varphi} - \varrho \frac{\epsilon^{abc} p_b n_c}{(p, n)^2} \dot{\varphi} \ . \tag{3.13}$$

The rest equations turn out to be identities under the above equations supplemented by the constraints. This could be expected, since the anyon possesses as many degrees of freedom as the spinless particle.

There exist some global restrictions on the world lines, which are related with causal requirements. Really, the causal conditions

$$\dot{x}^2 < 0 \ , \quad \dot{x}^0 > 0 \tag{3.14}$$

fulfil on the mass shell if and only if the following inequalities take place

$$s - \sqrt{s^2 + \varrho^2} < \frac{\varrho^2}{m} \frac{\dot{\varphi}}{(\dot{x}, n)} < s + \sqrt{s^2 + \varrho^2} \ , \tag{3.15a}$$

for $\varrho \neq 0$ and

$$\frac{2s}{m} \frac{\dot{\varphi}}{(\dot{x}, n)} < 1 \ , \tag{3.15b}$$

for $\varrho = 0$. It is interesting that Lagrangian (3.12) is well defined only if all the conditions are fulfilled.

Let us describe the general solution of the equations of motion under the gauge condition $\dot{x}^0 = 1$. As the Lagrangian has two independent gauge symmetries, this gauge condition leaves an arbitrary function $\varphi(t)$ to enter the general solution. The explicit form of the solution is as follows:

$$\vec{x}(t) = \frac{\vec{p}}{p^0} \left[t + m s \int_{\varphi(0)}^{\varphi(t)} \frac{d\varphi}{(p, n)^2} - \varrho \int_{\varphi(0)}^{\varphi(t)} d \left(\frac{1}{pn} \right) \right] + \tag{3.16}$$

$$+ m s \int_{\varphi(0)}^{\varphi(t)} \frac{\vec{n} d\varphi}{(p, n)^2} - \varrho \int_{\varphi(0)}^{\varphi(t)} d \left(\frac{\vec{n}}{p, n} \right) + \vec{x}(0) \ ,$$

where

$$\vec{n} = (-\cos\varphi, \sin\varphi) \qquad \vec{p} = const \qquad p^0 = \sqrt{\vec{p}^{\,2} + m^2}\,.$$

Thus one can see from (3.16) that, similar to the four-dimensional model [22] (and also in accordance with the 3d-analysis of the paper [19]), the uniform velocity motion of the particle $\vec{x}(t) = \vec{p}t/p^0$ is superimposed by the Zitterbewegung with the amplitude of the order of the de Broglie wave length. This Zitterbewegung can be eliminated by an appropriate choice of the frame being fixed by an appropriate gauge condition, e.g.: $\varphi(t) = \varphi(0)$, unlike the case of $d = 1 + 3$, where the Zitterbewegung is not a pure gauge artifact [22].

Now we would like to discuss the relationship of our model with some known anyon models. For $\varrho = 0$ there exists, in the Hamiltonian formulation, a local canonical transformation to twistor variables of the form

$$z\,, p_z \to q^\alpha = z^\alpha \sqrt{-2\varsigma p_z}\,, \qquad \{q^\alpha, q^\beta\} = \varsigma\epsilon^{\alpha\beta}\,, \tag{3.17}$$

where $\varsigma = sign\, s$. As a result, the action (3.5) can be locally transformed into the twistor action of 'semions' [20]:

$$S = \int d\tau \left\{ p_a \dot{x}^a - \frac{\varsigma}{2} q_\alpha \dot{q}^\alpha - \frac{e(\tau)}{2}(p^2 + m^2) + \lambda(\tau)\left(\frac{\varsigma}{4}(p^a\sigma_a)_{\alpha\beta}q^\alpha q^\beta + ms\right)\right\} \tag{3.18}$$

At the same time, the internal phase spaces, $T^*(S^1)$ and the real plane parametrized by the twistor variables q^α, have different topological structures. Thus the models (3.5) and (3.18) are not globally equivalent. This becomes most transparent at the quantum level where the spin spectrum turns out to be discrete for the Sorokin-Volkov model (semions) [20] and continuous for our model. Let us now consider a global canonical transformation such that the Pauli-Lubanski function W appears as one of the canonical variables

$$W = (p, n)p_\varphi - \varrho(p, \partial_\varphi n) \qquad \Psi = \int\limits_{\varphi_0}^{\varphi} \frac{d\varphi}{(p, n)} \tag{3.19}$$

$$\tilde{p}_a = p_a \qquad \tilde{x}^a = x^a - W\int\limits_{\varphi_0}^{\varphi} \frac{n^a d\varphi}{(p, n)^2} - \varrho\int\limits_{\varphi_0}^{\varphi} \frac{\epsilon^{abc}p_b n_c}{(p, n)^2}d\varphi\,.$$

In other words, we describe the spin in terms of action-angle variables: W is strongly conserved and Ψ is a pure gauge degree of freedom. Originally these variables were used by Plyushchay [18] to parametrize the phase space of the minimal model derived from an extended one by reducing the dynamics to special constrained surface. But such a reduction was accompanied by the loss of manifest covariance. Our consideration shows that the manifest covariance can be restored by passing to the variables z, p_z we use from the outset.

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3.2 Quantization

Let us consider canonical quantization of the model in the framework of the Dirac method [29]. This normally implies to perform the following. All the phase space variables should be defined as operators subject to the canonical commutation relations in a Hilbert space, while the physical state subspace is extracted by imposing the condition that physical wave functions should be annihilated by the operators of the first-class constraints. It is necessary also to supply the physical subspace with a well-defined inner product.

   In the present model, it is naturally used to realize the Hilbert space of one-particle states as a space of scalar fields on the configuration manifold $\mathcal{M}^4 = \mathbf{R}^{1,2} \times S^1$. Then the Poincaré group generators are realized in momentum representation as:

$$\mathcal{P}_a = p_a \qquad \mathcal{J}_a = i\epsilon_{abc}p^b\frac{\partial}{\partial p_c} - in_a\partial_\varphi + sn_a + \varrho\partial_\varphi n_a \qquad (3.20)$$

Now we can realize the first-class constraints as operators and define the physical states $\Psi(p,\varphi)$ by the following equations

$$(p^2 + m^2)\Psi(p,\varphi) = 0 \qquad (3.21)$$

$$((p,\mathcal{J}) - ms)\Psi(p,\varphi) = 0. \qquad (3.22)$$

For any two physical states Ψ_1 and Ψ_2 the Poincaré-invariant scalar product is defined by

$$< \Psi_1|\Psi_2 >= N\int\limits_{-\infty}^{\infty}\frac{d^2\vec{p}}{p^0}\int\limits_{0}^{2\pi}\frac{d\varphi}{(p,n)}\overline{\Psi_1(p,\varphi)}\Psi_2(p,\varphi), \qquad (3.23)$$

with $p^0 = \sqrt{\vec{p}^{\,2} + m^2}$ and N a normalization constant.

   We obtain the well-known realization of the fractional spin representations [18, 19]. The one-particle wave function of anyon (up to the factor of $(pn)^{1/2}$) is transformed on S^1 by the irreducible unitary representation of the principal continuous series of group $\overline{SO^\uparrow(1,2)}$ [25, 26]. Thereby the choice of the representation weight should be co-ordinated with the particle spin. The space of representation is infinite-dimensional and the constraint equation (3.22) has the meaning of the projection onto the corresponding one-dimensional subspace. Our approach is also agreement with the results of Ref. [18] where the quantization has been fulfilled with the use of action-angle-type variables (3.19).

   The equations (3.21) and (3.22) can be explicitly resolved as follows

$$\Psi(p,\varphi) = C(\vec{p})exp\left\{-2is\arctan\left(\frac{m\epsilon_{\alpha\beta}z^\alpha z_0^\beta}{p_{\alpha\beta}z^\alpha z_0^\beta}\right) - is\varphi - i\varrho\ln\frac{(p,n)}{(p,n_0)}\right\} \qquad (3.24)$$

where

$$p^0 = \sqrt{\vec{p}^{\,2} + m^2} \qquad p_{\alpha\beta} \equiv (p^a\sigma_a)_{\alpha\beta} \qquad z^\alpha = (1,\text{ctg}\frac{\varphi}{2}),$$

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and φ_0 is the integration constant of Eq. (3.22).

Let us point out that the generators $\mathcal{J}_a$ in (3.20) are chosen so that the wave functions (3.24) are well-defined on $\mathcal{M}^4$. There are possible different realizations for J_a. In particular, the third term in the expression (3.20) for J_a can be removed by an unitary transformation [19]. However the resulting wave function

$$\tilde{\Psi} = e^{is}\Psi \tag{3.25}$$

turns out to be multivalued on S^1.

It is worth discussing the relationship between the quantization of the minimal model for $\varrho = 0$ and the model of 'semions' [20] described by the action (3.18). These mechanical models are related at the classical level through the local canonical transformation (3.17), but their phase spaces have different topology. That is why the models possess different spin spectra. The quantization of the model (3.18) has been realized in Ref. [20] on the bounded below representations of the discrete series with lowest weights $1/4$ and $3/4$. Only the particles with spin $s = (2n+1)/4$, n being a positive integer, (so called semions) are allowed to appear in spectrum of model [20].

Finally, we consider in more detail the description of the case of (half) integer spin in the framework of our geometric construction. In that case the wave function of anyon (3.24) can be expanded on S^1 in relativistic harmonics of the form $z^{\alpha}/(p\xi)^{1/2}$. For spin $s = \pm k/2$ ($k = 0, 1, 2\ldots$) one gets

$$\tilde{\Psi} = exp(-i\varrho\ln(pn))F_{\alpha_1\alpha_2\ldots\alpha_k}(p)\frac{z^{\alpha_1}z^{\alpha_2}\ldots z^{\alpha_k}}{(p,\xi)^{k/2}} \ ,$$

and Eqs. (3.21,3.22) are equivalent to the Dirac equation for $F_{\alpha_1\alpha_2\ldots\alpha_k}(p)$

$$(p_{\alpha_1}{}^{\beta} + im\varsigma\,\delta_{\alpha_1}{}^{\beta})F_{\beta\alpha_2\ldots\alpha_k}(p) = 0 \tag{3.26}$$

$$F_{\alpha_1\alpha_2\ldots\alpha_k} = F_{(\alpha_1\alpha_2\ldots\alpha_k)} \qquad |s| = \frac{k}{2} \qquad \varsigma = sign\,s \ .$$

Thus, the (half) integer spin particle admits the description in terms of usual finite-component fields transforming by the finite-dimensional representation of $SL(2, \mathbf{R})$.

4   The anyon model on $T^*(\mathbf{R}^{1,2}) \times \mathcal{L}$

We turn now to the case $C < 0$. Then the spin space has to be homeomorphic to Lobachevsky plane. So the anyon dynamics can be realized on the phase space $T^*(\mathbf{R}^{1,2}) \times \mathcal{L}$.

Let us start with the following action functional

$$S = \int (p_a dx^a + \varrho\mathbf{\Omega}(z, \bar{z}) - H(x^a, p_a, z, \bar{z}, e, \lambda)d\tau) \tag{4.1}$$

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$$\mathbf{\Omega}(z, \bar{z}) = \frac{i}{\chi}(dz + d\bar{z}),$$

where the Hamiltonian

$$H = e(\tau)T_1 + \lambda(\tau)T_2$$

presents a linear combination of the constraints

$$T_1 = p^2 + m^2 \approx 0 \qquad T_2 = 2i\varrho\frac{(p, \zeta)}{\chi} - ms \approx 0. \tag{4.2}$$

Here $e(\tau)$ and $\lambda(\tau)$ are Lagrange multipliers. It follows from (4.1) that the symplectic structure on the manifold is determinated by the Poincaré-invariant two-form

$$\mathrm{d}p_a \wedge \mathrm{d}x^a + \varrho\mathbf{\Sigma} \tag{4.3}$$

with $\mathbf{\Sigma} = \mathrm{d}\mathbf{\Omega}$ a Poincaré-invariant two-form on the Lobachevsky plane. Due to (4.3), we are able to identify the phase space with $T^*(\mathbf{R}^{1,2}) \times \mathcal{L}$. The non-vanishing Poisson brackets read

$$\{x^a, p_b\} = \delta^a{}_b \qquad \{z, \bar{z}\} = \frac{i}{2\varrho}(z - \bar{z})^2. \tag{4.4}$$

It should be noted that the action functional (4.1) is not Poincaré-invariant since the one-form Ω changes on an exact one-form under an infinitesimal Lorentz transformation

$$\delta_\omega\Omega = \frac{i}{2}\omega^a\mathrm{d}(\partial\xi_a - \bar{\partial}\bar{\xi}_a). \tag{4.5}$$

To reveal the physical content of the model, we consider the Hamilton generators of Poincaré transformations

$$\mathcal{P}_a = p_a \qquad \mathcal{J}_a = \epsilon_{abc}x^b p^c + M_a, \tag{4.6}$$

$$M_a = 2i\varrho\frac{\zeta_a}{\chi} = -i\varrho\left(\frac{1 + z\bar{z}}{z - \bar{z}}, \frac{1 - z\bar{z}}{z - \bar{z}}, \frac{z + \bar{z}}{z - \bar{z}}\right). \tag{4.7}$$

Comparing the latter and (4.2) we show that the model describes an irreducible dynamic of the particle with mass m, spin s and timelike spin vector, $M_a M^a = -\varrho^2$.

The model on $T^*(\mathbf{R}^{1,2}) \times \mathcal{L}$ implies some restrictions on the parameters ϱ and s. Using the identity

$$u_a = 4\frac{(u, \zeta)}{\chi^2}\zeta_a - 2\frac{(u, \xi)}{\chi^2}\bar{\xi}_a - 2\frac{(u, \bar{\xi})}{\chi^2}\xi_a \tag{4.8}$$

for arbitrary 3-vector u_a, we get

$$u^2 = 4\frac{(u, \zeta)^2}{\chi^2} + 4\left|\frac{(u, \xi)}{\chi}\right|^2 \tag{4.9}$$

Now, applying the last identity for $\mu^a = p^a$ and accounting the constraints (4.2), we arrive at

$$s^2 - \varrho^2 = \left|\frac{2\varrho}{m}\frac{(p, \xi)}{\chi}\right|^2 \tag{4.10}$$

Thus the constraints (4.2) are non-contradictory only under the restriction $k = |s/\varrho| \geq 1$. Moreover, for $k = 1$ the momentum and the spin vector become parallel to each other

$$p_a = 2im\frac{\zeta_a}{\chi}\,. \tag{4.11}$$

As a consequence, for $\varrho = \pm s$ the structure of the constraints is drastically changed: instead of two first-class constraints we get two second-class constraints and one first-class constraint. In the following section we shall show that for $|k| = 1$ the model can be treated as a version of the well-known anyon model with Dirac's monopole two-form [11-14,19]

By eliminating p^a and the Lagrange multipliers from (4.1), we get the Lagrangian

$$L = -2m\sqrt{k^2 - 1}\left|\frac{(\dot{x}, \xi)}{\chi}\right| - 2imk\frac{(\dot{x}, \zeta)}{\chi} + i\varrho\frac{\dot{z} + \dot{\bar{z}}}{\chi}\,, \quad k = \frac{s}{\varrho}\,, \tag{4.12}$$

where the identity (4.9) has been used. From here we again get the restriction $|k| \geq 1$. For $|k| > 1$ the action possesses two gauge symmetries

$$
\begin{aligned}
\delta_{\epsilon_1} x^a &= 2p^a\epsilon_1 & \delta_{\epsilon_1} z &= \delta_{\epsilon_1}\bar{z} = 0\,; \tag{4.13}\\
\delta_{\epsilon_2} x^a &= 2i\varrho\frac{\zeta^a}{\chi}\epsilon_2 & \delta_{\epsilon_2} z &= -i(p, \xi)\epsilon_2 & \delta_{\epsilon_2}\bar{z} &= i(p, \bar{\xi})\epsilon_2\,.
\end{aligned}
$$

Here p^a is the three-momentum

$$p_a \equiv \frac{\partial L}{\partial \dot{x}^a} = -m\frac{\sqrt{k^2 - 1}}{|\chi|}\left[\sqrt{\frac{(\dot{x}, \bar{\xi})}{(\dot{x}, \xi)}}\,\xi_a + \sqrt{\frac{(\dot{x}, \xi)}{(\dot{x}, \bar{\xi})}}\,\bar{\xi}_a\right] - 2imk\frac{\zeta_a}{\chi}\,.$$

For $|k| = 1$, when $p_a = 2im\zeta_a/\chi$, $(p, \xi) = (p, \bar{\xi}) = 0$, the symmetries become dependent. This is a consequence of the fact that for $|k| = 1$ there remains only one first-class constraint.

Let us give a few comments on the dynamics. Similarly to the model on $\mathcal{M}^4$, here in general position $|k| > 1$ we have the Zitterbewegung and generalized causality conditions. The case $|k| = 1$ is again very special. Here and only here the Zitterbewegung is absent: the particle moves along a straight line in $\mathbf{R}^{1,2}$ and remains in rest on $\mathcal{L}$, the position on $\mathcal{L}$ being determined by the constraints.

5 Coupling to external fields

It is well known that for constrained dynamical systems the existence of consistent coupling to external fields is not obvious and deserves special study. The behavior of the 3d-spinning particle in arbitrary external electromagnetic and gravitational fields has been recently studied in the framework of the model with the Dirac monopole

symplectic two-form [11, 12]. In this approach the dynamics of a particle with mass m and spin s is realized in a six-dimensional phase space with the symplectic structure

$$\mathrm{d}p_a \wedge \mathrm{d}x^a + \frac{s}{2} \frac{\epsilon^{abc} p_a \mathrm{d}p_b \wedge \mathrm{d}p_c}{(-p^2)^{3/2}} \tag{5.1}$$

such that the mass-shell equation (3.3) presents itself the only constraint in the theory. As a consequence, the model admits coupling to an arbitrary background. From (5.1) one deduces the fundamental Poisson brackets

$$\{x^a, x^b\} = -s \frac{\epsilon^{abc} p_c}{(-p^2)^{3/2}} \qquad \{x^a, p_b\} = \delta^a{}_b \qquad \{p^a, p^b\} = 0. \tag{5.2}$$

The peculiar feature of the model is that the momentum and the spin vector are parallel. This is the same condition that is characteristic of the minimal anyon model for $|k| = 1$, when the model contains three constraints instead of two ones in the general position. For $|k| = 1$ and only in this case the model possesses one first-class and two second-class constraints, instead of two first-class constraints in general position. Obviously, only the former constraint's structure is stable with respect to deformations by weak background fields. In fact, for $|k| = 1$ the minimal model can be treated as a reformulation of the monopole model (5.1,5.2) in special extended phase space. This can be seen by reducing the dynamics on the surface of the second-class constraints.

In general position, the model seems to possess a limited number of admissible backgrounds. We present below two special cases of external fields. First, let us consider a minimal coupling of the model to a curved gravitational background. Then, the constraints turns into

$$T_1 = g^{mn} \nabla_m \nabla_n + m^2 \approx 0 \qquad T_2 = e_a{}^m \nabla_m M^a - ms \approx 0 \,, \tag{5.3}$$

where

$$\nabla_m = p_m + \omega_{m\,ab} \epsilon^{abc} M_c \,,$$

$\omega_{m\,ab}$ being the torsion-free spin connection and $e_m{}^a$ a dreibein associated to the metric g_{mn}. However, not every background field preserves the structure of the constraints. Using the identities

$$\{\nabla_a, \nabla_b\} = \left(\eta_{ab} R_{bd} + \eta_{bd} R_{ac} - \frac{1}{2} R \eta_{ac} \eta_{bd}\right) \epsilon^{cde} M_e \,,$$

where $R_{ab} = e_a{}^m e_b{}^n R_{mn}$, $R = R_a{}^a$ are Ricci tensor and scalar curvature respectively, one can arrive at the following commutation relation

$$\{T_1, T_2\} = -2 \epsilon^{abc} M_a R_{bd} M^d e_c{}^m \nabla_m. \tag{5.4}$$

This expression vanishes if and only if the space-time has a constant curvature, that is $R_{mn} = \frac{1}{3} R g_{mn}$. The constraints (5.3) could also be obtained as a dimensional reduction of the $d = 1 + 3$ anti de Sitter spinning particle model [23].

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Next, we consider the interaction of anyon with an external $U(1)$ - gauge field, using the following covariant generalization of momentum [27]:

$$p_a \to P_a = p_a + eA_a(x) + lF_a(x) \,,$$

where $A_a(x)$ is the Abelian gauge potential, $F_a(x) = 1/2\epsilon_{abc}F^{bc} = \epsilon_{abc}\partial^b A^c$ is the dual field strength, e and l are the charge and anomalous magnetic momentum of particle respectively. Then the Poisson bracket of the constraints

$$T_1 = P^2 + m^2 \approx 0 \qquad T_2 = (P, M) - ms \approx 0$$

transforms to

$$\{T_1, T_2\} = 2P_a M_b \epsilon^{abc}(eF_c + l\partial^d F_{cd}) \,. \tag{5.5}$$

It is seen that the bracket of constraints is proportional to the free equations of Chern-Simons topological massive electrodynamics, with e/l the mass of the field. The r.h.s. of (5.5) should vanish to preserve the constraints structure. In this way the free equations of the Chern-Simons electrodynamics emerge from the anyon model as compatibility conditions of the particle constrained dynamics. The topological mass of the field appears to be consistent with charge and anomalous magnetic momentum ratio for the anyon. The similar phenomenon is recognized in some superparticle and superstring models, where the massless field equations are reproduced for the *external fields* from the consistency requirements of the model dynamics [28].

In the case when the particle spin is not parallel to the momentum, one can see from (5.4,5.5) that the homogeneous gravity background and the free Chern-Simons Abelian gauge field probably exhaust all the backgrounds, allowing the consistent coupling to the anyon.

6   Concluding remarks

In this paper the minimal model of anyon has been considered in depth. Although the action of the $d = 3$ Poincaré group is nonlinearly realized on the spin phase space $(T^*(S^1)$ or $\mathcal{L})$, the suggested geometric construction provides the manifestly covariant formulation without any auxiliary variables introduced. Besides the formulation has the transparent geometric sense, it can give an efficient tool for the study of the quantization and coupling of the anyon to external fields. It should be noted that these problems are allowed to be treated by means of this construction without employing the known methods such as induced representations and Dirac-monopole two-form [10-14], which can not be applied in general case, when the particle momentum is not parallel to the spin vector.

We restricted our consideration of the quantum theory to the case of the phase manifold $T^*(\mathbf{R}^{1,2} \times S^1)$. Here the quantization of the minimal model naturally leads to the well known realization of quantum anyon theory on the fields transforming

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by the unitary representations of the continuous series of $\overline{SO(1,2)}$. The quantization on the manifold $T^*(\mathbf{R}^{1,2}) \times \mathcal{L})$ could be performed in a similar way. The only essential distinction is that the realization of the unitary representation of the Poincare-group requires to use the appropriate infinite-dimensional representation of the *discrete* series of Lorentz group. It is remarkable that the spinning part (4.7) of the Lorentz generators (4.6) coincides with the Berezin's symbols of generators of the corresponding representations used by geometric quantization on $\mathcal{L}$ [30, 26]. Thus, for the quantization in the spin subspace one can use the well-studied method of the geometric quantization.

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